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Colored non-Gaussian noise driven systems: Mean first passage time

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Abstract. We have examined the mean first passage time ($\langle T \rangle$) for a particle driven by colored non-Gaussian noise. As we depart from the Gaussian behavior, the $\langle T \rangle$ decreases regularly to a limiting value, *i.e.*, the barrier crossing rate can be accelerated to a limiting value by increasing the non-Gaussianity of the noise. For the non-Gaussian noise driven process $\langle T \rangle$ increases linearly with increasing damping constant or noise correlation time. But this increasing behavior is almost exponential in nature for the Gaussian noise driven process.

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Random processes are ubiquitous in almost all areas of physics, chemistry, biology etc. Among the large number of instances that one can name are the firing of neurons, the nucleation of a phase associated with a phase transition, the triggering of an alarm, the occurrence of a major earthquake, and the crossing of an activation barrier by a reaction coordinate that converts reactants to products in a chemical reaction, etc. These examples ranging from the macroscopic to the microscopic, indicating that the arrival problems are of interest at all scales. In many cases it is not only important to know the statistics of occurrence of such events, but more specifically the statistics of the first occurrence. This then leads to the study of the statistical properties of the time that it takes a random process to reach a specified state for the first time, that is, the mean first passage time (MFPT). The inverse of the MFPT is Kramer's rate [1] of barrier crossing dynamics. The literature for both is enormous and extends over many decades [2–10].

An important phenomenon in the random processes is the stochastic resonance (SR). It has attracted enormous interest due to two important aspects. First, SR has potential technological applications for optimizing the response to weak external signals in nonlinear dynamical systems. Second, it has connection with some biological mechanisms. So extensive work has been done in this particular field [11–13], which show the large number of applications in science and technology, ranging from paleoclimatology [14,15], to electric circuits [16], lasers [17], chemical systems [18,19] and the connection with some situations of biological interest (noise-induced information flow in sensory neurons in living systems, influence in ion-channel gating or in visual perception) [20–22].

Now it is important to note that a majority of such studies on the mean first passage time and the SR, have been done considering that the noise was Gaussian. However, some experimental results in sensory systems, particularly for one kind of crayfish [23] as well as recent results for rat skin [24], offer strong indications that the noise source in these systems could be non-Gaussian. Another recent study on neural networks also points in this direction [25]. Recent detailed studies on the source of fluctuations in some biological systems clearly indicate that both noise sources in general are non-Gaussian and their distributions are bounded [26]. However, very recently Fuentes *et al.* [27] showed that the stochastic resonance can be enhanced when the subsystem departs from Gaussian behaviour and the system shows marked "robustness" against noise tuning, *i.e.*, the signal-to-noise ratio curve can flatten when departing from Gaussian behaviour, implying that the system does not require fine tuning of the noise intensity in order to maximize its response to a weak external signal. This theoretical finding was verified experimentally by Castro *et al.* [28]. Fuentes *et al.* [29] also studied the effect of non-Gaussian noise on MFPT in the over damped limit. The objective of the present paper is to enquire whether the non-Gaussian noise can play a significant role in the context of mean first passage time in the weak damping region.

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To begin with we consider a stochastic process in the phase space. The relevant Langevin equation of motion can be written as

$$\dot{x} = p \tag{1}$$

$$\dot{p} = -V'(x) - \gamma p + \eta. \tag{2}$$

Here x and p correspond to position and momentum of the triggered particle. $V(x)$ in equation (2) is the potential energy of the particle. For the present problem we choose $V(x)$ as

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2. \tag{3}$$

To consider the effect of dissipation γ is used in equation (2). The last term in the equation (2) is due to external colored non-Gaussian random force. Time evolution of the random force η is given by

$$\dot{\eta} = -\frac{\eta}{\tau(1+\alpha(q-1)\eta^2/2)} + \frac{1}{\tau}\zeta(t) \quad . \tag{4}$$

Parameter q is used in the above equation to consider the deviation from Gaussian characteristic of the noise. Clearly, when $q \to 1$ we recover the limit of η being a Gaussian colored noise. Here τ is the noise correlation time and $\zeta(t)$ corresponds to the Gaussian white noise which has the following properties:

$$\langle \zeta \rangle = 0 \tag{5}$$

and

$$\langle \zeta(t)\zeta(t') \rangle = 2D\delta(t-t') \tag{6}$$

D is the noise strength. α in equation (4) is given by

$$\alpha = \frac{\tau}{D}\cdot \tag{7}$$

Thus equations (1–2) and equation (4) correspond to the case of diffusion of a particle in a potential $V(x)$, induced by η, a colored non-Gaussian noise. The stationary probability distribution for the stochastic variable η and $q > 1$ is given by [28,29]

$$P_q^{st}(\eta) = \frac{1}{Z_q}[1+\alpha(q-1)\eta^2/2]^{-1/(q-1)}, \tag{8}$$

with $\eta(-\infty,\infty)$ and Z_q in the above equation refers to the normalization factor having value

$$Z_q = \{\frac{\pi}{\alpha(q-1)}\}^{1/2}\frac{\Gamma(1/(q-1)-1/2)}{\Gamma(1/(q-1))}\cdot \tag{9}$$

Here Γ indicates the Gamma function. Although the above distribution function extends to $\pm\infty$, it does not exist for $q \geq 3$ (since Z_q diverges for $q \geq 3$) while the second moment diverges for $q \geq 5/3$ [29]. However, for $q < 1$ we have a cut-off distribution function [29] of the following form

$$P_q^{st}(\eta) = \begin{cases} \frac{1}{Z_q}[1-(\frac{\eta}{w})^2]^{1/(1-q)} & \text{if } |\eta| < w, \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

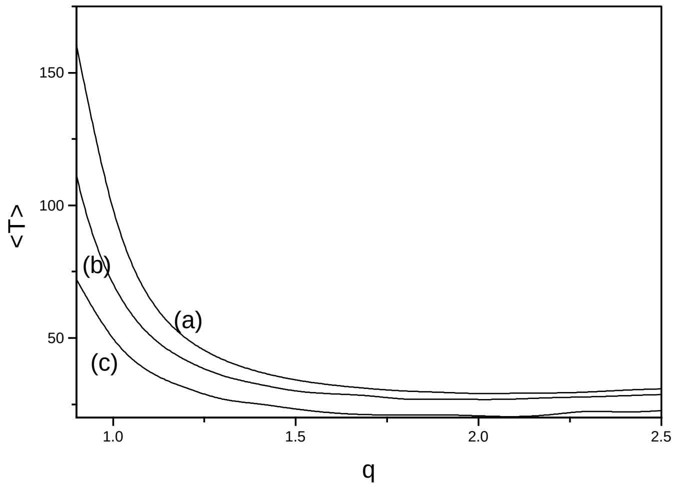

Fig. 1. Plot of the mean first passage time $\langle T \rangle$ *vs.* q for the parameter sets: (a) $\tau = 1.0$, $\gamma = 0.6$ (b) $\tau = 1.0$, $\gamma = 0.5$, (c) $\tau = 0.75$, $\gamma = 0.5$. $D = 0.1$ is used for all the curves. (Units are arbitrary.)

with the cut-off value given by $w = [(1-q)\alpha/2]^{1/2}$ and the normalization factor being

$$Z_q = \{\frac{\pi}{\alpha(q-1)}\}^{1/2}\frac{\Gamma(1/(q-1)+1)}{\Gamma(1/(q-1)+3/2)}\cdot \tag{11}$$

Here one more important point to be noted to distinguish the regions between $q \geq 1$ and $q < 1$ is that the correlation function of η in the stationary regime can be fitted by an exponential decay for $q \leq 1$, while for $q > 1$ it is fitted by Tsallis exponential [29].

We are now in a position to calculate the mean first passage time (MFPT). Here we consider the diffusion of a particle in a double well potential (3). The mean first passage time of interest is the time it takes the particle to go from one of the potential minima to the other when the transition is driven by the non-Gaussian fluctuations. To calculate $\langle T \rangle$ we solve equations (1–2) and equation (4) simultaneously using Heun's method. In our simulations we follow the dynamics of each particle starting in the left well at $x(t=0) = -1.0$ until it arrives in the right well at $x(t=T) = 1.0$ for the first time. We then calculate $\langle T \rangle$, that is, the average of T over many (say, 5,000) realizations. To examine how the departure of Gaussian characteristic of the color noise effects the $\langle T \rangle$, we have calculated $\langle T \rangle$ for different q. The results are shown in Figure 1. This calculation implies that the mean first passage time decreases with increasing non-Gaussian behavior ($q > 1$) of the noise and ultimately reaches a limiting value for a given set of values of noise strength, noise correlation time and damping constant. Thus the barrier crossing rate is accelerated by increasing deviation from Gaussian characteristic of the colored noise up to certain value. This is due to the fact that with increasing non-Gaussian behavior($q > 1$) effective noise strength and noise correlation time are increased (denominator of the first term of the right hand side of Eq. (4) increases with an increase of q) [29]. Since the diffusion of a particle is accelerated

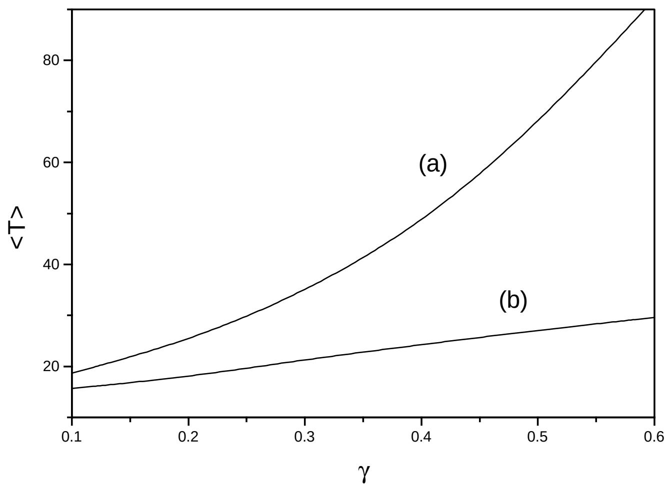

Fig. 2. Plot of $\langle T\rangle$ *vs.* γ for $\tau = 1.0$, $D = 0.1$ and (a) $q = 1.0$, (b) $q = 1.9$. (Units are arbitrary.)

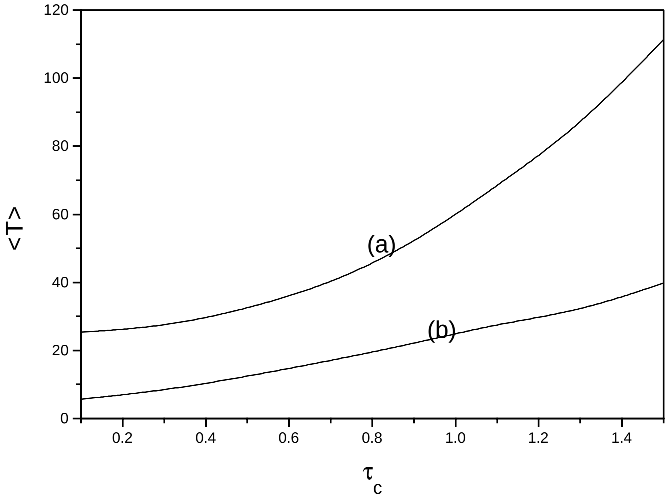

Fig. 3. Plot of $\langle T\rangle$ *vs.* τ for $\gamma = 0.5$, $D = 0.1$ and (a) $q = 1.0$, (b) $q = 1.9$. (Units are arbitrary.)

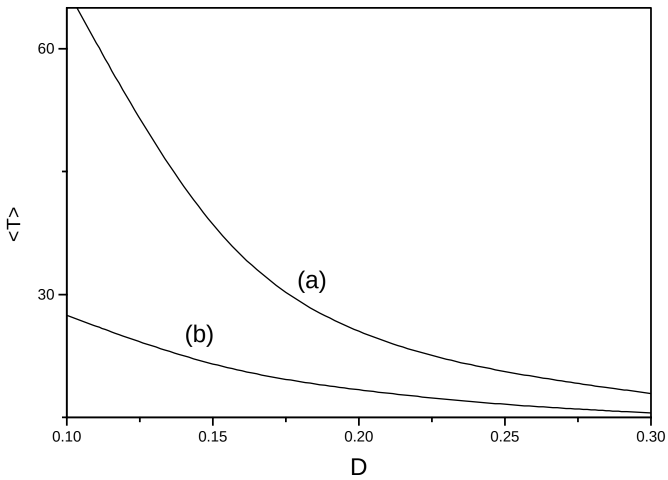

Fig. 4. Plot of $\langle T\rangle$ *vs.* D for $\gamma = 0.5$, $\tau = 1.0$ and (a) $q = 1.0$, (b) $q = 1.9$. (Units are arbitrary.)

by the former and retarded by the latter, for a large deviation from Gaussian behavior the interplay of these two quantities gives a limiting value of the MFPT. Therefore the limiting value of $\langle T\rangle$ is obtained at a higher value of q for smaller τ (curves (b) and (c) of Fig. 1) and for different values of the dissipation parameter γ at the same q for a given external noise strength (curves (a) and (b) of Fig. 1). However, for a large deviation from Gaussian behaviour ($q > 1$) understanding of the diffusion behaviour as well as MFPT and stochastic resonance of the non-Gaussian noise driven system is very difficult since the second moment and the correlation time of the external non-Gaussian noise diverge (for details see the Ref. [29]). In this regime our preliminary numerical experiment suggests that in some region of phase space the particle may be trapped for an infinite time or it escapes from some region of phase space within a very short time.

The effect of the damping constant γ on the $\langle T\rangle$ shows a significant difference for Gaussian and non-Gaussian noise driven processes. For the Gaussian noise driven process $\langle T\rangle$ increases almost exponentially with the increase of γ [29] but in the case of the non-Gaussian noise driven process $\langle T\rangle$ increases linearly as shown in Figure 2. Here both the Gaussian and non-Gaussian noise driven systems correspond to the thermodynamically open systems (noise and dissipation are not related through a fluctuation-dissipation relation). Although the external noise strength (D) is same for both the curves (a) and (b) of Figure 2, the effective noise strength is higher for non-Gaussian noise driven system and because of this MPFT increases more slowly with an increase of the damping constant γ than the Gaussian noise driven system. A similar feature is also found in the variation of the mean first passage time with noise correlation time τ and it is shown in Figure 3. Since the effective noise strength decreases more slowly with the increase of the noise correlation time for the non-Gaussian noise driven system than the Gaussian noise driven system by virtue of equation (4), $\langle T\rangle$ increases more slowly for the former system than for the latter system [30].

To examine the effect of noise strength D on $\langle T\rangle$ we have calculated it for different values of D. The results are illustrated in Figure 4. The decrease of $\langle T\rangle$ with an increase of D mimics the exponential decay as shown by Fuentes *et al.* [29] in the overdamped case. With the increase of the external noise strength D the effective noise strength as well as the effective noise correlation time are increased for the non-Gaussian noise driven system. Since noise strength and noise correlation time act on the diffusion of a particle in opposite directions, for large D the effect of increased noise strength is balanced by increased correlation time and hence with the increase of D the difference of MFPT for the Gaussian and non-Gaussian noise driven systems decreases as shown in Figure 4. Thus the interplay of γ and τ implies that the non-Gaussian noise is more effective than that for the Gaussian noise in the barrier crossing dynamics when the external noise strength is small.

In the present paper we have presented a numerical study for the first passage time problem in systems driven by non-Gaussian colored noise. We have considered the first passage of a particle that evolves in a potential from one well over the barrier to the other well. Our calculated results indicate that the mean first passage time decreases monotonically to a limiting value with increasing non-Gaussianity of the noise. The present calculation also implies that $\langle T \rangle$ increases linearly with the increase of damping constant or noise correlation time for the non-Gaussian noise driven process but the increasing behavior is almost exponential in nature in the Gaussian noise driven process. The present study also indicates that the difference of MFPT for Gaussian and non-Gaussian noise driven systems decreases with increase of the external noise strength. Since a number of recent studies have shown that the non-Gaussian noise driven systems are concerned in many situations in biology, we hope that our present observations will be useful support to the experimental findings.

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